SOLUTION STRATEGIES, MODES OF REPRESENTATION AND JUSTIFICATIONS OF PRIMARY FIVE PUPILS IN SOLVING PRE ALGEBRA PROBLEMS: AN EXPERIENCE OF USING TASK-BASED INTERVIEW AND VERBAL PROTOCOL ANALYSIS

Gan We Ling
Tun Abdul Razak Teachers' College
Sarawak, Malaysia

Munirah Ghazali
Universiti Sains Malaysia, Penang

This descriptive study was aimed at looking into how Primary 5 pupils solve pre-algebra problems concerning patterns and unknown quantities. Specifically, objectives of this study were to describe Primary 5 pupils' solution strategies, modes of representations and justifications in: (a) discovering, describing and using numerical and geometrical patterns, and (b) solving for unknown quantities in word problems. Subjects of this study consisted of four Primary 5 pupils from a rural primary school in Kota Samarahan, Sarawak. All the subjects were given a set of five pre algebra problems. The first problem concerned numerical pattern while the second and third problems concerned geometrical patterns. The fourth and fifth problems were word problems involving unknown quantities. Pupils were asked to write down all the steps used in solving those problems and at the same time verbalise their thinking. Data were collected through pupils' verbal think aloud protocols, retrospective questioning and observation. All verbal data were transcribed before they were analyzed. All written work by pupils was also analysed. Findings of this study suggested that pupils displayed different solution strategies and used various modes of representation to solve problems concerning patterns. In solving for unknown quantities in word problems, pupils justified their strategies differently even though their solution strategies appeared similar. Major errors made in the process of solving those pre algebra problems were also discussed. Some problems and suggestions to improve the use of task-based interview and collection of verbal think aloud protocol were discussed at the end of this paper.

Background of the Study

In Malaysia, algebra may seem to be a very strange word in the minds of the primary school pupils. This is not surprising at all since algebra is most probably taught formally and directly to the pupils in the classroom. In fact, mathematics under the New Primary School Curriculum (*Kurikulum Baru Sekolah Rendah* or *KBSR*) actually "contains" some elements of algebra. For instance, finding missing addend, minuend or subtrahend in arithmetic equations is actually algebraic as it involves the process of organizing the arithmetic needed to find an answer to a question involving quantities that are not yet known. Choike (2000) defined this process as "algebra".

Statement of the Problem

This study was undertaken to see how Primary 5 pupils solve pre-algebra problems. The pre-algebra problems used in this study focused on (a) recognising, generalising and justifying numerical and geometrical patterns, and (b) using the arithmetic needed to work with unknowns in the form of word problems. Considering that the pupils have no prior formal and direct exposure to algebra, the study proposes to study how they would use their prior knowledge and experiences in mathematics to solve pre-algebra problems in terms of their solution strategies and modes of representation used, and how they would justify their solution processes?

Purpose of the Study

This study intended to describe explain how Primary 5 pupils discover, explain and use numerical and geometrical patterns. The patterns used in this study were growing patterns that involve a progression from step to step. These problems may require pupils to extend, explore and perhaps look for a generalisation or an algebraic relationship that will tell them what the pattern will be at any point along the way. For problems involving unknown quantities, this study would describe how Primary 5 pupils use mathematical operations to solve for unknowns presented in the form of word problems. Their process of solving those pre-algebra problems was studied with respect to their solution strategies, modes of representation and justifications.

Research Questions

This study intended to answer the following research questions:

a) What are the solution strategies, modes of representation and justifications of Primary 5 pupils to discover, describe and use 1) numerical patterns, and 2) geometrical patterns?

b) What are the solution strategies, modes of representation and justifications of Primary 5 pupils to solve for unknown quantities in word problems?

Some Related Literature Review

Solution Strategy

According to Anderson (1987), cognitive psychologists distinguished two types of cognitive strategies. The first type is general cognitive strategies for problem solving such as brainstorming, means-end analysis, reasoning through analogy, the use of worked examples, working backward and working forward. These strategies can be applied to problems in many different domains. The second type of cognitive strategy is domain-specific strategies such as looking for a pattern, which may only be applied to problems in a particular domain such as mathematics, particularly algebra.

Modes of Representation

These are the external representations of students' solution processes which reflect their mathematical thinking (Cai, 1995). Cai (1995) classified modes of representation into verbal (spoken or written words), visual (picture or drawing), arithmetic symbolic (use of numbers) and algebraic symbolic representations. Examination of these modes of representation revealed the ways in which pupils solve problems and reflected the ways in which pupils communicate their mathematical ideas and thinking processes.

Mathematical Justification

In solving mathematical problems, pupils could be asked to evaluate the reasonableness of their answers and solution processes, make and evaluate mathematical conjectures and arguments and validate their own thinking. McCoy, Baker and Little (1996) stressed that students actually seek understanding when they conjecture, argue and justify their solution processes.

Pre-Algebra

Pre-algebra concerns recognising, generalising and justifying patterns which involves constructing various representations (Friedlander & Hershkowitz, 1997). It also involves understanding number system in order to work with unknowns or variables and properties of operations (Urquhart, 2000).

Operation Sense

Arithmetic cannot be separated with operations. Slavit (1999) explained that the ability to use arithmetical operations as "operation sense". He then further elaborated that "operation sense involves various kinds of flexible conceptions" (p.254) about the underlying structure and use of mathematical operations as well as relationships among these operations. Schifter (1999) also explained that when the children come to see that any missing addend problem can be solved by subtraction, they evidence a sense of how the operations are related and acquired experience with the inverse relationships of addition and subtraction. This is related to MacGregor and Stacey's (1999) view that ability to see the reasons behind relationships requires a generalisation about properties of numbers and this ability is deeply algebraic.

Limitations of the Study

This study involved collecting and analysing verbal think aloud protocols during task-based interviews. According to Cai (1995), "the process of collecting, coding and analysing verbal protocol data is extremely labour intensive" (p.7). So, a large sample was not feasible for this study. Thus, the results of this study were indicative and were confined only to the subjects chosen for this study.

Methodology

Design of the Study

This descriptive study took the design of a case study as only one primary school was involved.

Subjects of the Study

The subjects of this study consisted of four Primary 5 pupils from one rural primary school in the Samarahan Division of Sarawak. The selection of subjects were based on one criterion that the subjects need to be able to articulate verbally well due to the data collection method chosen for this study.

Data Collection

Verbal think aloud protocols. Since this study involved knowledge elicitation, techniques like process tracing through "think aloud" method became the main means of data collection. Verbal protocol was carried out concurrently and retrospectively. Concurrent protocol was done by asking subjects to solve the pre-algebra problems and at the same time asked them to verbalise their thinking. According to Ericsson and Simon (1980,1984) (cited in Hassebrock & Prietula, 1992), concurrent verbalisation provided the most complete report since information was verbalised as processing and verbalisation occurred at the same time and therefore, no thought, feeling, or action would be omitted because the participant had no time to forget! This added to the validity of this method in collecting data about thinking processes. Sometimes, retrospective questioning was done after concurrent protocol as a supplement to provide the missing information or to fill the gaps in concurrent protocol. Collection of verbal think aloud protocols yielded information about the knowledge and thought processes that underlie observable task performance (Chipman, Schraagen, & Shalin, 2000). In this way, the thought processes underlying subjects' solution processes and justifications could be collected through these verbal think aloud protocols.

Task-based interview. Task-based interview is a research instrument for making systematic observations in the psychology of learning mathematics and can be adapted as assessment tools for describing the subject's knowledge (Goldin, 2000). It focuses research attention more directly on the subject's process of addressing mathematical tasks, rather than just on the patterns of correct and incorrect answers in the results produced.

Observation. The primary focus of observation in this study was the subjects' order of solution processes and modes of representations used in solving the pre algebra problems.

Instrument

A set of five pre algebra problems were administered to the subjects to explore their strategies used. These problems allowed the subjects to produce and display their process used to obtain an answer. This provided visible record of their solution processes and use of representations. These five pre-algebra problems were taken and adapted from a few studies (e.g. Kaput & Blanton, 2001; Femiano, 2003; and Larkin, Perez, & Webb, 2003). Problem 1 required subjects to discover, describe and use numerical pattern whereas Problems 2 and 3 involve geometrical patterns. Problems 4 and 5 required the subjects to solve for the unknown quantities in the word problems. These problems were used as they covered the scope of pre algebra as discussed in literature review.

Analysis of Data

Verbal protocol analysis. The first step in analysing a verbal protocol was to break down the transcript into short segments or phrases. This step in analysing the protocol yielded a topic representation in which each segment addressed a particular instance of reasoning behavior on the task. The topic representations were then coded based on pupils' solution strategies, modes of representation, justifications and errors (if there were) to describe the cognitive processes subjects used to solve pre algebra problems.

Content analysis. Besides verbal protocol analysis, documentary and content analysis was conducted. Content analysis involves identification and classification of content (Anderson, 1998). It was used in this study to analyse the written solution process and modes of representation used by subjects in their written work.

Execution of the Study

Considering that the main source of data for this study came from subjects' verbal data, the rapport with the subjects was very important in order that data can be elicited from them. To achieve this, the subjects were introduced to the researcher through their teacher. The researcher emphasised to all the subjects that the solution process, not the answer, was more important in this study. Assurance of confidentiality and anonymity was also stressed to every subject before the collection of verbal think aloud protocols.

Every session of the task-based interview was audio taped and transcribed for each subject. Subjects' verbal think aloud protocols was analyzed task by task according to the subjects' solution strategy, mode of representation and justification. This paper will only present the analysis of findings.

Analysis of Findings

Problem 1 (Numerical Pattern)

What is the number to be filled in the box following the number sequence below:
87, 81, 75, 69, $\square$

Solution strategy. All the subjects seemed able to discover the existence of a common difference between consecutive terms in the number sequence in Problem 1 without much difficulty. Three methods were identified in finding the common difference - finding the difference between two consecutive numbers (e.g. "87 - 81" or "75 - 69"), counting back (86, 85…81) and mental subtraction (87 - ? = 81). All of them used "69 - 6" to arrive at the answer.

*Mode of representation***.** Only arithmetic symbolic representation in the form of standard algorithms was used to solve this problem and verify the answer.

*Justification***.** All the subjects were able to justify the use of subtraction operation in the process of finding the common difference as well as to get the required numerical answer. They elicited the schemata for subtraction due to the decreasing value of terms in the numerical pattern.

Problem 2 (Irregular Geometrical Pattern)

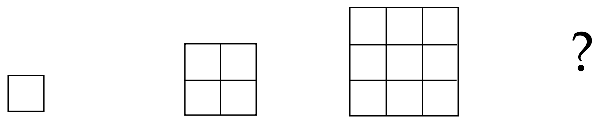


Figure 1 Figure 2 Figure 3 Figure 5

By referring to Figure 1, Figure 2 and Figure 3, draw Figure 5.

Solution strategy. Three of the subjects seemed to discover and described the geometrical pattern in terms of the number of rows and columns. To solve the problem, they first drew a big square with 16 squares in 4 rows and 4 columns, then continued to draw another big square with 25 squares in 5 rows and 5 columns and called it Figure 5. Another subject solved this problem by constructing Figure 4 based on Figure 3 by adding 7 squares in an inverted "L" shape, thus yielding a bigger square with 16 squares in 4 rows and 4 columns. In the similar way, he constructed Figure 5 based on Figure 4.

Mode of representation. All the subjects used verbal and visual representations to solve this problem and justify their answer. They preferred to explain their solution process and justification verbally in their own words with the aid of diagrams. Figure 1 below shows the solution process and visual representation used by one of the subjects who solved this problem differently from the other three subjects.

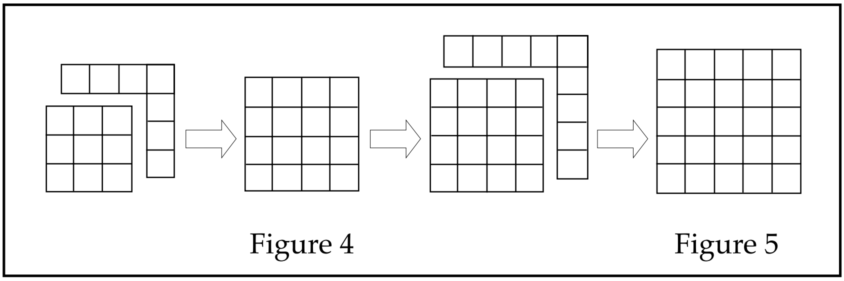


Figure 1. One subject's solution process in solving Problem 2.

Justification. Justification was based on the increasing number of rows and columns. When asked to justify their answer, three subjects generalized that "Figure 1 has 1 row and 1 column; Figure 2 has 2 rows and 2 columns; Figure 3 has 3 rows and 3 columns; Therefore, Figure 5 must have 5 rows and 5 columns". Another justification was based on the "construction" method. The sole subject who used a different solution process explained and drew how Figure 2 was constructed from Figure 1. He then explained and drew how Figure 2 was constructed to form Figure 3.

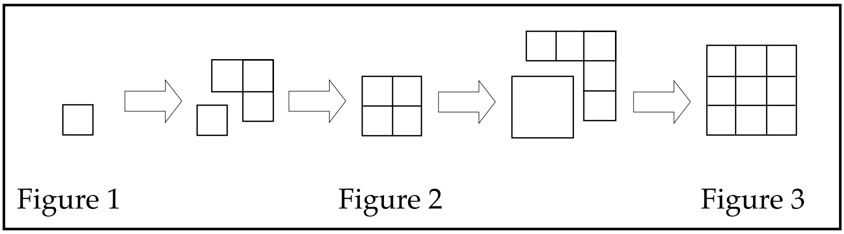
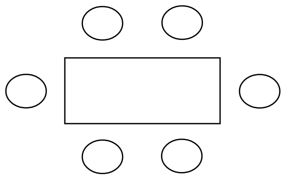


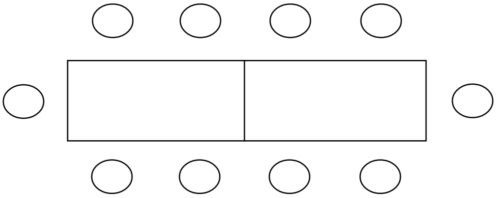
Figure 2. One subject's discovery of pattern & justification for Problem 2.

Problem 3 (Regular Geometrical Pattern)

Ali is arranging some tables for his birthday party. 6 persons can be seated around a table as below:



When Ali puts two tables end to end, 10 persons can be seated around the table as below:



How many persons can be seated around 3 tables which are put end to end?

Solution strategy. Two subjects seemed to discover the pattern by comparing the first and second diagrams. They did not describe the pattern verbally but produced the required diagram almost instantaneously. Two other subjects seemed to relate the two diagrams to a numerical pattern that reflected the number of persons seated around the table (6, 10,…). All the subjects arrived at the answer through drawing with reference from previous drawings. One subject constructed his diagram based on the diagram showing 2 tables with ten "persons" seated around it. He shifted the "person" sitting at the width of the second table to the width of the third table and put four "persons" on the lengths - two on each side of the length of the third table. The other subjects drew a diagram consisted of three tables arranged end to end, then seated 14 "persons" around the three tables to get the answer.

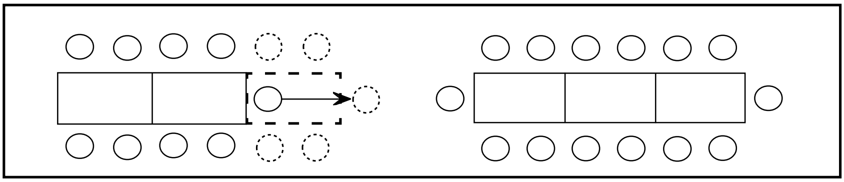


Figure 3. Solution strategies and visual representation used by subjects for Problem 3

Mode of representation. All the subjects used visual representation to solve the problem. However, two of them used verbal and numerical representation to verify their answer.

Justification. The justification was based on extension of diagram – "follow the diagram before this" as said by the subjects. Two subjects used numerical pattern with a common increment in value to verify their answer. One of the subjects justified her verification by pointing at the diagram with one table and said "six", then she pointed at the diagram with two tables arranged end to end, she said "ten" and mentioned "six plus four is ten". Finally she said "so ten plus four is fourteen".

Problem 4 (Single-step Word Problem involving Unknown)

Bahtiar has read some books. If he reads 5 more books, the total number of books
he reads will become 17 books. How many books has Bahtiar read before this?

Solution strategy. All the subjects used subtraction operation to get the answer. Some examples of subjects' verbal protocols that suggested their line of thought were: "number of books read is 17, including 5 more books" and "he needs 5 more books to make up 17". Analysis of these verbal protocols seemed to suggest they were thinking along "? + 5 = 17" to solve this problem.

Mode of representation. Only arithmetic symbolic representation involving the use of standard algorithms was used to solve this problem and verify the answer.

*Justification***.** Subtraction operation was justified differently. One subject based on the word "before this". Another subject said "because he needs 5 more books to make it 17, so 17 minus 5". Two subjects said "12 plus 5 is 17, so I used 17 minus 5 to get 12". These two subjects verified their answer by using addition.

Problem 5 (Multiple-step Word Problem involving Unknowns)

Mariam will be 20 years old in 3 years. Her brother, Dahlan's age is 2 years more than Mariam. What is Dahlan's age now?

Solution strategy. Three subjects used subtraction and addition operations while another one used counting back and then counting on to solve the problem. Two subjects used subtraction to obtain Mariam's age and used addition to get the answer. One used mental subtraction to get Mariam's age and wrote "17 + 2 = 19" in standard algorithm as the answer. Another one used the counting back method ("19, 18, 17") to get Mariam's age and then counting on ("18, 19") in finding Dahlan's age.

Mode of representation. Only arithmetic symbolic / numerical representation was used in solution process. Three of the subjects performed subtraction ("20 – 3") and addition ("17 + 2") operations in standard algorithms.

Justification. The subjects explained that subtraction operation and counting back method was used to get Mariam's present age due to the phrase "in 3 years" whereas the addition operation and counting on method was justified through being "2 years more". This seemed to suggest that the subjects used the "keyword" problem-solving method.

Discussion of Findings

Numerical Pattern

The subjects in this study seemed to be able to discover the common difference from one number to the previous or next one in the numerical pattern. They were able to describe the trend of numerical pattern and then generate the following number from the previous number based on the common difference identified in the number pattern. Verification of the required answer was done by using the common difference. The subjects tended to use numerical representation only in discovering, describing and extending numerical pattern.

Geometrical Patterns

For the problem involving irregular or growing geometrical pattern (Problem 2), the subjects seemed to discover and describe the pattern differently. Three of the subjects who described the pattern in terms of number of rows and columns had the difficulty in generating the required figure. Moreover, the problem required them to generate the fifth figure based on first, second and third figure. Their generalization was incomplete and they overlooked the existence of the fourth figure. However, they were still able to justify their solution process verbally with the aid of diagrams.

One subject exhibited his ability to discover and describe the next figure as being the extension of the previous figure. He constructed the required figure without any difficulty and was able to justify his solution process correctly. Could this be due to the way he discovered and described the pattern in a "constructive" method that led him to the answer easily compared to the other subjects?

Problem 3 involved a regular geometrical pattern. All subjects were able to recognize the pattern "hidden" in the diagrams though the time taken ranged from instantaneously to pause for 30 seconds. Two subjects seemed to be able to relate the visual representation used in the problem to numerical representation in justifying their answer.

Unknown Quantities in Word Problems

Problem 4 was a single-step problem involving missing addend as interpreted by the subjects. Two subjects' made their efforts to verify their answer. Using subtraction operation to find the missing addend and then used addition operation to verify the subtraction operation reflected the subjects' ability of "operation sense" as mentioned by Slavit (1999). These two subjects seemed to be able to understand and apply the inverse relationships between addition and subtraction operations. This finding also seemed to be in line with what Schifter (1999) said – when the children come to see that any missing addend problem can be solved by subtraction, they evidenced a sense of how the operations are related and acquired experience with the inverse relationships of addition and subtraction.

Problem 5 was a multiple-step problem. Two of the subjects voiced their same confusion whether to find Mariam's age now or three years later. Could this be due to the existence of multiple unknowns in this problem that challenged their ability of comprehending the problem? Subjects used both formal (arithmetical and numerical) and informal (counting on and counting back) methods in solving this problem. In terms of their justifications, they seemed able to explain why subtraction and addition operations, or counting back and counting on were used to achieve the sub goal and goal of the problem. However, their reliance on the keywords in the problem may lead to them learning a set of rote operations based on the keywords without necessarily understanding the semantic structure of the problem.

Conclusions

Based on the analysis and discussion of findings, some conclusions could be drawn for this study.

All the subjects used domain-specific strategies, particularly "look for a pattern" to solve problems concerning patterns. For word problems most of the subjects preferred to use formal strategies particularly arithmetical and numerical strategies involving numbers and operations. General problem solving strategies like "working forward", "working backward", "identifying sub goal" and "drawing diagram" were also used.

Verbal and arithmetic symbolic/numerical representation seemed to be the most commonly used modes of representation in the solution process, including verification of answers in almost all problems. Visual representation was used only in problems involving diagrams, particularly geometrical patterns. Algebraic symbolic representation was not used directly at all!

The subjects seemed to be able to justify their solution processes and answers verbally for pre-algebra problems concerning numerical patterns and unknown quantities in word problems. For problem concerning geometrical patterns, verbal justifications were made with the aid of diagrams. Justifications made by the subjects seemed limited to explanations about their solution processes and verify their answers. All the subjects did not go further, to argue and make conjectures.

Since mathematical errors "is a natural part of mathematics reasoning" (Bruning, Schraw & Ronning, 1995, p. 340), it is thus not uncommon to detect some errors made by the subjects during data collection. Inaccurate generalisation was the major error found in problem involving irregular or "growing" geometrical pattern (Problem 2). Misinterpretation and misunderstanding of problem seemed to happen for Problem 2 and Problem 4 as these they were rather indirect in nature. No major errors were found for the other problems, except carelessness and mechanical use of operations that led to incorrect answer.

Implications of this Study

Findings of this study indicated that the subjects were able to solve pre-algebra problems to a certain extent. They exhibited their ability to use the "look for a pattern" strategy to solve problems involving regular patterns. They took a shorter time to solve problems concerning regular numerical pattern compared to regular geometrical or pictorial pattern. Could this reflect that teachers and textbooks present problems numerically much more frequent than geometrically?

Some subjects displayed their ability to make connection among two different modes of representation. According to Driscoll and Moyer (2001), the ability to make connections among different representations is one indicator of algebraic thinking – a process of identifying and extending patterns (Van De Walle, 2001). Some subjects also seemed to possess certain ability in "operation sense". Slavit (1999) argued that operation sense can be transitioned into algebraic ways of thinking. Do these abilities imply subjects' emerging ability to think algebraically, as mentioned by Cai (1998)? However, findings of this study indicated that Primary 5 pupils did not use algebraic representation in solving pre algebra problems. Does this imply that algebraic approach can be considered to be introduced in the teaching and learning of mathematics in the primary schools, particularly in problem solving?

Recommendations

The researchers feel that algebraic ways of thinking do not need to be postponed until later stages of schooling. As pointed out by Urquhart (2000), understanding number system and working with properties of operations are some algebraic skills that can be developed early. Primary school pupils can also be encouraged to engage in algebraic activities like recognise, describe, extend, create and generalize growth patterns, which is actually at the heart of mathematics as a science of pattern and order. These skills may develop into algebra which focuses on the generalization of patterns as well as to reason, explain and justify (Friedlander & Hershkowitz, 1997).

It is also recommended that task-based interview be adapted as a classroom-based instrument in assessing how pupils solve mathematical problems. This would the enable teachers to observe their pupils' use of solution strategies and ask their pupils to justify their solution processes used. While making justifications, pupils learn to argue, conjecture and evaluate the reasonableness of their answers. Greenes & Findell (1999) also added that by requiring students to document in writing or describe orally their thinking and justify their solutions, they are not only develop their communication skills but also their understanding and facility with the language of mathematics in general and algebra in particular.

Problems and Suggestions

The researchers would like to share some problems faced in the use of task-based interviews and verbal think aloud protocols as the main means of data collection. This method of data collection required the subjects to verbalize what were they thinking about while solving the pre-algebra problems. Most of the time, the subjects were so involved in the problem-solving process and fell into complete silence. In addition, they seemed to be very much aware of the presence of the tape recorder. So the researcher had to remind them not only to talk but also to talk aloud so that their voice could be recorded.

As a result, the researcher often used retrospective questioning. Questions such as "what were you thinking?", "how did you do this?", "why did you do this?", "how do you know this is true?" were commonly asked by the researcher to the subjects in the process of data collection. Simon & Kaplan (1989) warned that during retrospective questioning, subjects might reconstruct their lines of thought that did not actually happen while performing the task. Thus care must be taken cautiously while analysing retrospective protocol data.

To overcome this problem, perhaps the researcher can prepare one or two simple problems to act as "warming up" for the subjects to help them get used to verbalising their thinking while solving the problems. Through informal discussion with all the subjects, they admitted that it was the first time they had to verbalise their thinking while solving a problem and justify their solution process after solving the problem!

Videotaping could have been used in the data collection process. In this study, some subjects tended to shake or nod their heads instead of saying "no" or "yes". They also tended to use their fingers to aid their counting. These gestures might provide some kind of non-verbal data but could not be recorded by audio recording.

Since the subjects may have not much prior experience dealing with geometrical patterns, manipulatives such as counters and tiles could be provided to help them in solving Problems 2 and 3. Goldin (2000) explained that this would permit the subjects to exercise a range of possibilities in representing their responses observably. Thus, inferences about internal representations or mental processes can be drawn from the subjects' external manipulations and productions.

Note: This paper first presented at the International Conference on Science and Mathematics Education (CoSMEd) 2005 on the theme of 'Bridging the Theory-Practice Gap in Science and Mathematics Education: The Challenge to Change'. It focuses on the Conference Sub-theme of 'Bridging the Theory-Practice Gap through Innovations in Curriculum and Assessment.'

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